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EXPECTED NUMBER OF CROSSINGS OF AN ARBITRARY CURVE BY A NON-STATIONARY GAUSSIAN PROCESS

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SUMMARY

We ~~derive~~ a formula for the expected number of crossings of an arbitrary, possibly discontinuous, curve in a time interval $(0,T)$ by a continuous non-stationary normal process. It is shown that a formula analogous to that of M. R. Leadbetter and J. D. Cryer [3] holds under conditions more general than they considered.

Introduction

We consider the expected number of (non-tangential) crossings of an arbitrary, possibly discontinuous, curve in a time T by a continuous non-stationary normal process. It is shown that a formula analogous to that derived by M. R. Leadbetter and J. D. Cryer, holds under conditions more general than they considered in [3]. The method used is that developed by N. Donald Ylvisaker in [4]. For the sake of simplicity we assume that the process and the curve have been simultaneously normalized so that we may consider crossings of a curve $\nu(t)$ by a continuous process with mean function 0 and with variance 1. It is shown that if $\nu(\cdot)$ is of unbounded variation, then the expected number of crossings is $+\infty$.

The proofs are somewhat more intuitive than those of [3] in that they proceed directly from a calculation of the probability involved instead of an interplay of Dirac delta functions and the process derivative.

The formula for the expected number of crossings is a sum of two parts; the first is just an integral of Rice's formula, the derivation of which is independent of the second part and follows Ylvisaker, [4]. For $\nu(\cdot)$ = constant this gives a slightly more general form of Ylvisaker's proof of the necessary and sufficient conditions for Rice's formula.

The second part of the sum represents the expected number of crossings of the vertical component of the curve. If the curve is a step function,

the second part gives the sum of the probabilities of crossing the vertical portions of the curve. Examination of this part can give information about the error involved if we replace the curve with a step function, (thereby simplifying the necessary calculations).

As a corollary we get a simplified form of the result of Leadbetter and Cryer under conditions somewhat more general than they consider. The simplification in form is essentially a result of the prior normalization of the process and curve to get a process with variance function 1. If this normalization is made, the hypothesis in [3] that $\Gamma_{12}(t,t')$ is continuous at diagonal points is equivalent to our hypothesis that the $\sigma(t)$ of Theorem 1 is continuous.

The expected number of crossings of a curve of unbounded variation has little practical application, but the fact that it is always $+\infty$ for continuous Gaussian processes does give information about the behavior of such process; and for non-Markovian processes very little such information is available in the literature.

Notation

Given a point function $\nu(\cdot)$ of bounded variation, let $\nu\{\cdot\}$ be the measure determed by:

$$\nu\{(o,t]\} = \nu(t) - \nu(o).$$

Let the measure $|\nu|\{\cdot\}$ be the total variation of $\nu\{\cdot\}$. Let $|\nu|(\cdot)$ be the point function determined by $|\nu|\{\cdot\}$:

$$|\nu|(t) = |\nu|\{(o,t]\}.$$

We shall say that $X(t)$ has crossed the curve $\nu(t)$ in the time interval T if there exists t_1, t_2 in T such that $(X(t_1) - \nu(t_1)) \cdot (X(t_2) - \nu(t_2)) < 0$. The number of crossings of the curve is then the maximum number of disjoint time intervals in which $X(\cdot)$ crosses $\nu(\cdot)$.

Theorem 1:

Suppose $X(t)$ is a non-stationary Gaussian process with mean function 0 and variance function 1.

We assume that the correlation function $\rho(t_1,t_2) = EX(t_1)X(t_2)$ has the spectral form

$$\rho(t_1,t_2) = \int_{-\infty}^{\infty} \cos 2\pi\lambda(t_2-t_1)F(t_1,d\lambda)$$

with the extended real valued continuous on $(0,T)$ second spectral moment

$$\sigma^2(t) = \int_{-\infty}^{\infty} (2\pi\lambda)^2 F(t,d\lambda).$$

Let $N(T)$ denote the number of crossings of a fixed curve $\nu(t)$ by the process $X(t)$.

Then:

$$EN(T) = +\infty$$

if $\nu(\cdot)$ is of unbounded variation, and if $\nu(\cdot)$ is of bounded variation:

$$(1) \qquad EN(T) = \int_0^T \frac{\sigma(t)}{\pi} e^{-\frac{1}{2}\nu^2(t)} dt + \int_0^T \psi(\sigma,\nu,t)\,|\nu|\{dt\}$$

where

$$\psi(\sigma,\nu,t) = 2\phi(\nu(t)) \int_0^1 dp \int_0^{p|\nu'(t)|/\sigma(t)} \phi(y)\,dy$$

at continuity points of $\nu(\cdot)$

$$= 2\int_0^1 \phi(p\nu(t^+) + (1-p)\nu(t))\,dp$$

at left continuous discontinuity points of $\nu(\cdot)$,

$$= 2\int_0^1 \phi(p\nu(t^-) + (1-p)\nu(t))\,dp$$

at right continuous discontinuity points of $\nu(\cdot)$,

$$= 2\int_0^1 [\phi(p\nu(t^+) + (1-p)\nu(t)) + \phi(p\nu(t^-) + (1-p)\nu(t))]\,dp$$

at points of total discontinuity of $\nu(\cdot)$.

<u>Concerning the Formula (1)</u>

The first integral in (1) is just the integral of Rice's formula for the number of crossings of a fixed level; it does not involve $\nu'(\cdot)$.

If $\nu(\cdot)$ is constant, then the integral with respect to $|\nu|(\cdot)$ vanishes.

On the other hand, if $\nu(\cdot)$ is a step function, then the second integral is the sum of the probability that $X(t)$ crosses the various vertical components of $\nu(\cdot)$. In this sense it may be said that the first term represents the expected number of crossings of the horizontal component of $\nu(\cdot)$ and the second term represents the expected number of crossings of the vertical component.

<u>Proof of Theorem 1</u>:

Let $d = \{t_i: 0 < t_1 < t_2 < \cdots < t_{n_d} < T\}$ be a collection of points of the interval $(0,T)$. Let $y_d(t)$ be the stochastic process whose graph is a series of straight lines with vertices at $(x(t_i) - \nu(t_i), t_i)$. Let $N_d(T)$ be the number of axis crossings by $y_d(t)$.

We shall be interested in taking limits over some fixed sequence of collections $\{d_i\}_1^\infty$ having the property that d_{i+1} contains d_i, and

$$\lim_{i\to\infty} \max_{t_k, t_{k+1} \in d_i} (t_{k+1} - t_k) = 0.$$

For the portion of the proof dealing with $\nu(\cdot)$ of bounded variation, we further assume that $\bigcup_1^\infty d_i$ contains all the discontinuity points of $\nu(\cdot)$.

Let $N_d(T)$ be the number of axis crossings by $y_d(t)$, $0 \le t \le T$. Then:

$$N(T) \ge N_d(T)$$

and if $\nu(\cdot)$ is of bounded variation

$$\lim N_d(T) = N(T) \qquad \text{a.s.}$$

hence

$$\lim EN_d(T) = EN(T).$$

Taking expectations:

$$\begin{aligned} EN_d(T) &= P_r\{y_d(t) \text{ has an axis crossing in } (t_i,t_{i+1})\} \\ &= \Sigma P_r\{X(t_{i+1}) > \nu(t_{i+1})\} - P_r\{X(t_{i+1}) > \nu(t_{i+1}),\ X(t_i) > \nu(t_i)\} \\ &\quad + P_r\{X(t_i) > \nu(t_i)\} - P_r\{X(t_i) > \nu(t_i),\ X(t_{i+1}) > \nu(t_{i+1})\}. \end{aligned}$$

Let $L(h,k,\rho)$ be the probability that $Z > h$, $W > k$, when Z and W have a bivariate normal distribution with means 0, variances 1, and correlation ρ.

Then

$$\begin{aligned} EN_d(T) = \Sigma\{&L(\nu(t_{i+1}),\ \nu(t_{i+1}),\ 1) - L(\nu(t_{i+1}),\ \nu(t_i),\ \rho(t_{i+1},t_i)) \\ &+ L(\nu(t_i),\ \nu(t_i),\ 1) - L(\nu(t_i),\ \nu(t_{i+1}),\ \rho(t_{i+1},t_i))\} \end{aligned}$$

$$= \Sigma\left\{\frac{1}{t_{i+1}-t_i}[L(\nu(t_{i+1}),\ \nu(t_{i+1}),\ 1) - L(\nu(t_{i+1}),\ \nu(t_{i+1}),\ \rho(t_{i+1},t_i))]\right\} \cdot (t_{i+1} - t_i)$$

$$+ \Sigma\left\{\frac{1}{t_{i+1}-t_i}[L(\nu(t_i),\ \nu(t_i),\ 1) - L(\nu(t_i),\ \nu(t_i),\ \rho(t_{i+1},t_i))]\right\} \cdot (t_{i+1} - t_i)$$

$$+ \Sigma\left\{\frac{1}{|\nu|(t_{i+1})-|\nu|(t_i)}\left[\int_{\nu(t_i)}^{\nu(t_{i+1})}\int_{\nu(t_i)}^{\nu(t_{i+1})} \phi(x,y,\rho(t_{i+1},t_i))dxdy\right]\right\} \cdot (|\nu|(t_{i+1}) - |\nu|(t_i))$$

$$= \int_0^T n_d^{(1)}(t)dt + \int_0^T n_d^{(2)}(t)dt + \int_0^T n_d^{(3)}(t)|\nu|(dt)$$

where $n_d^{(1)}(t)$, $n_d^{(2)}(t)$, $n_d^{(3)}(t)$ are the step functions represented in the braces in the above sums.

The proof of the theorem follows from Lemmas 1, 2, 3, and 4.

To simplify the notation let t be an element of the collection d_i (and hence an element of all subsequent d_i), and let τ be the decreasing difference from t to its successor in d_i. We will abbreviate $\rho(t,t+\tau)$ by $\rho(\tau)$.

<u>Lemma 1</u>. Under the conditions of Theorem 1, if $\nu(\cdot)$ is of bounded variation, then:

$$\lim \int_0^T n_d^{(1)}(t)dt = \frac{1}{2\pi} \int_0^T e^{-\frac{1}{2}\nu^2(t)}\left[\int_0^\infty (2\pi\lambda)^2 F(t,d\lambda)\right]^{\frac{1}{2}}dt$$

in the sense that if one side of the equality is $+\infty$ then both sides are.

<u>Lemma 2</u>. Lemma 1 is true with $n_d^{(1)}(t)$ replaced by $n_d^{(2)}(t)$.

<u>Lemma 3</u>. Under the conditions of Theorem 1, if $\nu(\cdot)$ is of bounded variation, then

$$\lim \int_0^T n_{(d)}^{(3)}(t)|\nu|\{dt\} = \int_0^T \psi(\sigma,\nu,t)|\nu|\{dt\}$$

where

$$\psi(\sigma,\nu,t) = 2\phi(\nu(t))\int_0^1 dp\int_0^{p|\nu'(t)|/T(t)} \phi(y)dy$$

at continuity points of $\nu(\cdot)$,

$$= 2\int_0^1 \phi(p\nu(t^+) + (1-p)\nu(t)dp$$

at left continuous discontinuity points of $\nu(\cdot)$,

$$= 2\int_0^1 \phi(p\nu(t^-) + (1-p)\nu(t))dp$$

at right continuous discontinuity points of $\nu(\cdot)$,

$$= 2\int_0^1 [\phi(p\nu(t^+) + (1-p)\nu(t)) + \phi(p\nu(t^-) + (1-p)\nu(t))]dp$$

at points of total discontinuity of $\nu(\cdot)$.

This completes the proof of the theorem when $\nu(\cdot)$ is of bounded variation. Lemma 4 covers the remaining case where $\nu(\cdot)$ is of unbounded variation.

<u>Proof of Lemma 1</u>. Consider

$$K(\tau) = \frac{1}{\tau}[L(\nu(t+\tau), \nu(t+\tau), \rho(0)) - L(\nu(t+\tau), \nu(t+\tau), \rho(\tau))].$$

Using formula 3, page vi, of [1], we have

$$K(\tau) = \frac{1}{\tau}\left[\frac{1}{2\pi}\int_{\text{arc}\cos\rho(0)}^{\text{arc}\cos\rho(\tau)} \exp[-\nu^2(t+\tau)/1+\cos w]dw\right].$$

Then

$$K(\tau) \le \frac{1}{2\pi}\,\frac{\text{arc}\cos\rho(\tau)}{\tau}$$

and by the mean value theorem

$$K(\tau) = \frac{1}{2\pi}\, e^{-\frac{1}{2}\nu^2(t+\tau)}\left(\frac{\text{arc}\cos\rho(\tau)}{\tau}\right) + \frac{o(\tau)}{\tau}\,.$$

To evaluate the limit of $K(\tau)$, we use a series expansion for arc sin y:

$$\text{arc}\sin y = y + \frac{y^3}{6} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{y^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{y^7}{7} + \ldots \quad y^2 < 1,$$

and

$$\text{arc}\cos x = \text{arc}\sin\sqrt{1-x^2}.$$

Letting

$$A(\tau) = \frac{1-\rho^2(\tau)}{\tau^2} = (1+\rho(\tau))\int_0^\infty \frac{1-\cos 2\pi\lambda\tau}{\tau^2}\, dF(\lambda)$$

we have

$$\left[\frac{\text{arc cos }(\tau)}{\tau}\right] = A^{\frac{1}{2}}(\tau) + \frac{\tau^2 A^{3/2}(\tau)}{6} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\tau^4 A^{5/2}(\tau)}{5} + \ldots;\ \rho^2(\tau) > 0.$$

Therefore

$$\left[\frac{\text{arc cos}\rho(\tau)}{\tau}\right] \leq \left[2\int_0^\infty \left(\frac{1-2\cos 2\pi\lambda\tau}{\tau^2}\right) F(t,d\lambda)\right]$$

and

$$\lim_{\tau\downarrow 0}\left[\frac{\text{arc cos}\rho(\tau)}{\tau}\right] = \lim_{\tau\downarrow 0} 2\left[\int_0^\infty \left(\frac{1-\cos 2\pi\lambda\tau}{\tau^2}\right) F(t,d\lambda)\right]^{\frac{1}{2}}$$

in the sense that if one side of the equality is $+\infty$ then both sides are $+\infty$.

But the non-negative function $\frac{1-\cos 2\pi\lambda\tau}{\tau^2}$ is bounded above by and converges to $2\pi^2\lambda^2$. It follows that

$$K(\tau) \leq \frac{1}{2\pi}\left[\int_0^\infty (2\pi\lambda)^2 F(t,d\lambda)\right]^{\frac{1}{2}}$$

and

$$\lim_{\tau\downarrow 0} K(\tau) = \frac{1}{2\pi}\, e^{-\frac{1}{2}\nu^2(t^+)}\left[\int_0^\infty (2\pi\lambda)^2 F(t,d\lambda)\right]^{\frac{1}{2}}$$

in the sense that if one side of these equalities is $+\infty$ then both sides are $+\infty$.

The statement of Lemma 1 follows since $\nu(t) = \nu(t^+)$ a.e. ∎

The <u>Proof of Lemma 2</u> follows in the same way.

<u>Proof of Lemma 3</u>. If t_i is a point of discontinuity for $\nu(\cdot)$, it is a point of positive mass for $|\nu|\{\cdot\}$ and we must do better than to find the a.e. limit of $n_d^{(3)}(t)$ at such points. For the sake of

convenience we will assume that $t = t_i$ is a point of left continuity of $\nu(\cdot)$; the necessary modifications are recognizable if this is not the case.

Letting $\tau = t_{i+1} - t_i$; $\rho(\tau) = \rho(t,t+\tau)$:

$$n_d^{(3)}(t) = \frac{1}{|\nu|(t+\tau) - |\nu|(t)} \int_{\nu(t)}^{\nu(t+\tau)} \int_{\nu(t)}^{\nu(t+\tau)} \phi(x,y,\rho(\tau))\,dx\,dy$$

$$= \frac{1}{|\nu|(t+\tau) - |\nu|(t)} \int_{\nu(t)}^{\nu(t+\tau)} \phi(x)\,dx \int_{\frac{\nu(t)-\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}}^{\frac{\nu(t+\tau)-\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}} \phi(y)\,dy.$$

(That $n_d^{(3)}(t)$ is bounded above by a $|\nu|(\cdot)$ integrable function follows if we take 1 as an upper bound for the integral over y and $\phi(o)$ as an upper bound for the integrand of x.)

Substituting

$$x = p\nu(t+\tau) + (1-p)\nu(t), \quad 0 < p < 1,$$

let

$$M(\tau) = \frac{\nu(t+\tau) - \nu(t)}{|\nu|(t+\tau) - |\nu|(t)} \int_0^1 \phi(p\nu(t+\tau) + (1-p)\nu(t))\,dp \cdot \int_a^b \phi(y)\,dy$$

$$= \frac{|\nu(t+\tau) - \nu(t)|}{|\nu|(t+\tau) - |\nu|(t)} \int_0^1 \phi(p\nu(t+\tau) + (1-p)\nu(t))\,dp \cdot \int_{a\wedge b}^{a\vee b} \phi(y)\,dy$$

where

$$a = a(p,\tau) = \frac{\nu(t) - \rho(\tau)[p\nu(t+\tau) + (1-p)\nu(t)]}{\sqrt{1 - \rho^2(\tau)}}$$

$$b = b(p,\tau) = \frac{\nu(t+\tau) - \rho(\tau)[p\nu(t+\tau) + (1-p)\nu(t)]}{\sqrt{1 - \rho^2(\tau)}} .$$

To evaluate $\lim_{\tau \downarrow 0} a(\rho,\tau)$, write

$$a(\rho,\tau) = \frac{-p[\nu(t+\tau) - \nu(t)] + (1-\rho(\tau))[p\nu(t+\tau) + (1-p)\nu(t)]}{\sqrt{1 - \rho^2(\tau)}}$$

$$= -p\left(\frac{\nu(t+\tau)-\nu(t)}{\tau}\right) \cdot \frac{\tau}{\sqrt{1 - \rho^2(\tau)}} + o(\tau).$$

Since $0 < \int_0^\infty (2\pi\lambda)^2 F(t,d\lambda) < +\infty$, $\nu(\cdot)$ is twice differentiable, and we may evaluate $\lim \tau/\sqrt{1 - \rho^2(\tau)}$ by squaring and applying L'Hospital's rule:

$$\lim \frac{\tau^2}{1 - \rho^2(\tau)} = \frac{1}{-\rho''(o)}, = \frac{1}{\sigma^2(t)}$$

therefore

$$\lim_{\tau \downarrow 0} a(p,\tau) = -p\nu^+(t)/\sigma(t) \qquad \text{uniformly in } p, \ 0 \le p \le 1.$$

Similarly

$$\lim_{\tau \downarrow 0} b(p,\tau) = (1-p)\nu^+(t)/\sigma(t) \qquad \text{uniformly in } p, \ 0 \le p \le 1.$$

Thus

$$\int_{a\wedge b}^{a\vee b} \phi(y)\,dy = \int_{-p|\nu^+(t)|/\sigma(t)}^{(1-p)|\nu^+(t)|/\sigma(t)} \phi(y)\,dy.$$

It follows from the orthogonal decomposition of a signed measure that

$$\lim_{\tau \downarrow 0} \frac{|\nu(t+\tau) - \nu(t)|}{|\nu|(t+\tau) - |\nu|(t)} = 1.$$

Hence

$$M(o) = \int_0^1 \phi(p\nu(t^+) + (1-p)\nu(t))dp \int_{-p|\nu^+(t)|/\sigma(t)}^{(1-p)|\nu^+(t)|/\sigma(t)} \phi(y)dy$$

$$= \int_0^1 \phi(p\nu(t^+) + (1-p)\nu(t))dp \int_{-p|\nu^+(t)|/\sigma(t)}^{p|\nu^+(t)|/\sigma(t)} \phi(y)dy$$

∎

<u>If $\nu(\cdot)$ Is of Unbounded Variation</u>

<u>Lemma 4</u>. Suppose $\nu(\cdot)$ is a function of unbounded variation and $\sigma(t)$ is bounded or is extended real valued continuous.

Then

$$\lim_{\max(t_{i+1}-t_i)\to 0} \sum \int\int_{\nu(t_i)}^{\nu(t_{i+1})} \phi(x,y,\rho(t_{i+1},t_i))dxdy = +\infty$$

<u>Corollary</u>. It follows from Fatou's lemma and the inequality $N(T) \geq N_{(d)}(T)$ that if $\nu(\cdot)$ is of unbounded variation and $\sigma(t)$ is continuous or is bounded on $(0,T)$, then $EN(T) = +\infty$.

<u>Proof of Lemma 4</u>. As in Lemma 3, we have:

$$M(d) = \sum \int_{\nu(t_i)}^{\nu(t_{i+1})} \int_{\nu(t_i)}^{\nu(t_{i+1})} \phi(x,y,\rho(t_{i+1},t_i))dxdy =$$

$$\sum(\nu(t_{i+1}) - \nu(t_i)) \int_0^1 \phi(p\nu(t_{i+1}) + (1-p)\nu(t_i))dp \int_a^b \phi(x)dx$$

where a,b are as in Lemma 3. We may take a more conservative range of integration by setting

$$c_i(p) = 0 \vee \left(\frac{p|\nu(t_{i+1})-\nu(t_i)|}{\sqrt{1-\rho^2(t_{i+1},t_i)}} - \sqrt{\frac{1-\rho(t_{i+1},t_i)}{1+\rho(t_{i+1},t_i)}}\,|p\nu(t_{i+1}) + (1-p)\nu(t_i)| \right)$$

then

$$M(d) \geq 2\sum |\nu(t_{i+1}) - \nu(t_i)| \int_0^1 \phi\, dp \int_0^{c_i(p)} \phi\, dx.$$

Let $\hat{\nu} = \max |\nu(t)|$, $0 \leq t \leq T$, and fix ε, $0 < \varepsilon < 1$; then

$$M(d) \geq \varepsilon\phi(\hat{\nu})\sum |\nu(t_{i+1}) - \nu(t_i)| \cdot \int_0^{c_i(p)(1-\varepsilon)} \phi(x)dx.$$

Define the set S_d depending on $d = \{t_i\}$ as follows:

$$S_d = \left\{ i: \frac{|\nu(t_{i+1}) - \nu(t_i)|}{\sqrt{1 - \rho(t_{i+1},t_i)}} \geq \frac{1}{(1-\varepsilon)}(5+\hat{\nu}) \right\}$$
$$= \{ i: c_i(1-\varepsilon) \geq 5 \}.$$

Then for $i \in S_d$ we have

$$\int_0^{c_i(p)(1-\varepsilon)} \phi(x)dx \geq \int_0^5 \phi(x)dx = \delta,$$

$$M(d) \geq \varepsilon \cdot \delta \cdot \phi(\hat{\nu}) \sum_{S_d} |\nu(t_{i+1}) - \nu(t_i)|.$$

If $\nu(\cdot)$ is a function of unbounded variation, then this sum over *all* i is unbounded for any sequence of d having the property that

$$\lim \max_{t_i \in d} (t_{i+1}-t_i) \to 0;$$

thus Lemma 4 follows from Lemmas 5 and 6.

Lemma 5. The sequence $d_i = \{t_1, t_2, \ldots\}$ may be picked so that $\sum \sqrt{1-\rho^2(t_{j+1},t_j)}$ is bounded above for all i; and $\max_{d_i} (t_{j+1}-t_j) \to 0$.

Proof.

$$\frac{\sqrt{1-\rho^2(t_{j+1},t_j)}}{t_{j+1}-t_j} = \left[(1+\rho(t_{j+1},t_j))\int \frac{(1-\cos 2\pi\lambda(t_{j+1}-t_j))}{(t_{j+1}-t_j)^2} F_{t_j}\{d\lambda\}\right]^{1/2}$$

$$\leq \left[(1+\rho(t_{j+1},t_j))\int \frac{(2\pi\lambda)^2}{2} F_{t_j}\{d\lambda\}\right]^{1/2}$$

$$\leq \sigma(t_j).$$

If $\sigma(\cdot)$ is bounded on (o,T) we are through. If it is continuous and integrable, then for a given i, let the t_j $j = 1,\ldots,2^i$ in d_i be picked so that $\sigma(t_j) = \min \sigma(t)$, t in the interval $(j/2^i)T \leq t \leq (j+1/2^i)T$. Then

$$\sum \sqrt{1-\rho^2(t_{j+1},t_j)} < \sum \sigma(t_j)(t_{j+1}-t_j) \leq \int_0^T \sigma(t)dt. \quad \blacksquare$$

Lemma 6. Let $a_i^{(n)} \geq 0$ be a sequence such that

$$L_n \sum_1^{\infty} a_i^{(n)} = +\infty.$$

Let $b_i^{(n)} \geq 0$ be a sequence such that

$$\sum_1^{\infty} b_i^{(n)} \leq K, \text{ some fixed } K.$$

For arbitrary fixed M, let $S_n = \{i: a_i^{(n)}/b_i^{(n)} \geq M\}$.

Then:

$$L_n \sum_{S_n} a_i^{(n)} = +\infty.$$

<u>Proof</u>. Let $S_n^c = \{i: a_i/b_i < M\}$. We show $\sum_{S_n^c} a_i^{(n)}$ is uniformly bounded:

$$\sum_{S_n^c} a_i^{(n)} \leq \sum_{S_n^c} M\cdot b_i^{(n)} \leq M\cdot \sum_{S_n^c} b_i^{(n)} \leq M\cdot K. \quad \blacksquare$$

<u>Corollary</u>. (Leadbetter and Cryer) Suppose that the function $\nu(\cdot)$ (and hence the function $|\nu|(\cdot)$ and the measure $|\nu|\{\cdot\}$) are absolutely continuous with respect to Lebesgue measure. Then the formula (1) of Theorem 1 reduces:

$$(2) \qquad EN(T) = \int_0^T \left[\frac{\sigma(t)}{\pi} e^{-\frac{1}{2}\nu^2(t)} + \sigma(t)\phi(\nu(t))\int_0^{\nu'(t)/\sigma(t)} (2\Phi(y)-1)dy\right] dt.$$

<u>Proof</u>. In lieu of the absolute continuity of $|\nu|\{\cdot\}$ we may rewrite (1) as an integral with respect to Lebesgue measure; consider the integrand of the second portion of that integral:

$$n(t) = \psi(\sigma,\nu,t) \cdot |\nu'(t)|$$

$$= 2|\nu'(t)|\phi(\nu(t))\int_0^1 dp\int_0^{p|\nu'(t)|/\sigma(t)} \phi(y)dy ,$$

$$n(t) = 2\sigma(t)\phi(\nu(t))(\nu'(t)/\sigma(t))\int_0^1 dp\int_0^{p(\nu'(t)/\sigma(t))} \phi(y)dy.$$

Writing $\xi = \nu'(t)/\sigma(t)$, and considering $n(t,\xi) = n(t)$:

$$n(t,0) = 0$$

and

$$\frac{\partial n(t,\xi)}{\partial \xi} = \sigma(t)(2\Phi(\xi)-1).$$

Therefore,

$$n(t,\xi) = \sigma(t)\phi(\nu(t))\int_0^{\xi}(2\Phi(y)-1)\,dy.$$

The *corollary* is unlike the result of [3] only in that we have assumed $\nu(\cdot)$ absolutely continuous instead of continuously differentiable; if we consider a process with variance function 1, then the hypothesis of [3] that $\Gamma_{12}(t,t')$ exist and is continuous at diagonal points is exactly our requirement that $\sigma(t)$ exist and is continuous. To show that the formula of [3] reduces to (2) when the variance function is 1, note the typographical error on p. 510:

it follows (adding a circumflex to denote the notation of [3]) from Lemma 3 of [3] that

$$\hat{n}(t) = (\hat{m}'-\hat{\gamma}\hat{\rho}\hat{m})/[\hat{\gamma}(1-\hat{\rho}^2)^{\frac{1}{2}}].$$

If the variance is unity, the following simplifications may be made:

$$\hat{\sigma}^2 = 1$$

$$\hat{\gamma}^2 = \sigma^2(t)$$

$$\hat{\rho}^2 = 0.$$

With these substitutions the formula (2) of [3] may be reduced to (2) above by the same method of differentiating with respect to $\hat{m}'(t)/\sigma(t)$.

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